PERIOD-VI

M12CH20

TOPIC

20 Probability and Statistics

20.1. FUNDAMENTAL PRINCIPLE OF COUNTING (FPC)

The fundamental principle of counting (FPC) states that if an operation can be performed in 'm' different ways, following which another operation can be performed in 'n' different ways, then both operations, in succession can be performed in exactly 'm × n' different ways.

Working Steps

Step I: *Identify the independent operations involved in the given problem.*

Step II: *Find the number of ways of performing each operation.*

Step III: *Multiply these numbers to get the total number of ways of performing all the operations.*

Example 1. *In a class there are 25 boys and 15 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?*

Solution. No. of ways of selecting one boy out of 25 boys = 25

No. of ways of selecting one girl out of 15 girls = 15

25 Boys
15 Girls

$\therefore$ By **FPC**, total number of ways of selecting one boy and one girl = 25 × 15 = **375**.

Example 2. *How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?*

Solution A number between 100 and 1000 has 3 digits.

No. of ways of filling hundred's place = 2

($\because$ Either 2 or 9 is to be used)

No. of ways of filling ten's place = 2
No. of ways of filling unit's place = 2
∴ By **FPC**, total number of numbers
$= 2 \times 2 \times 2 = \mathbf{8}$.

Place:	H	T	U
	↓	↓	↓
Ways:	2	2	2

20.2. FACTORIAL NOTATION

Let $n \in \mathbf{N}$. The continued product of first n natural numbers (beginning with 1 and ending with n) is called **factorial** n and is denoted by $n!$

Thus, $\boldsymbol{n! = 1 \cdot 2 \cdot 3 \ldots\ldots (n-1) \cdot n}$

Illustrations $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$8! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 40320.$

Factorial zero is defined as equal to 1 and we write $0! = 1$.

It is easily seen that $n! = n \cdot (n-1)!$

$= n(n-1) \cdot (n-2)!$

$= n(n-1)(n-2) \cdot (n-3)!$

....

$8! = 8 \times 7! = 8 \times 7 \times 6! = 56 \times 6!$ etc.

Example 3. *Evaluate:*

(*i*) $\dfrac{7!}{5!}$ (*ii*) $\dfrac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{15!}$

(*iii*) $\dfrac{11!}{7!\,4!}$ (*iv*) $\dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!}$.

Solution. (*i*) $\dfrac{7!}{5!} = \dfrac{7 \times 6 \times 5!}{5!} = \mathbf{42}$.

(*ii*) $\dfrac{16 \times 15 \times 14 \times 13 \times 12!}{15!} = \dfrac{16(15 \times 14 \times 13 \times 12!)}{15!} = \dfrac{16(15!)}{15!} = \mathbf{16}.$

(*iii*) $\dfrac{11!}{7!\,4!} = \dfrac{11 \times 10 \times 9 \times 8 \times 7!}{7!\,(4 \times 3 \times 2 \times 1)} = \dfrac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2} = \mathbf{330}.$

(*iv*) $\dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!} = \dfrac{7 \times 6}{7 \times 6 \times 5!} + \dfrac{7}{7 \times 6!} + \dfrac{1}{7!} = \dfrac{42}{7!} + \dfrac{7}{7!} + \dfrac{1}{7!} = \mathbf{\dfrac{50}{7!}}$

Example 4. *If $x \in \mathbf{N}$, then solve the following equations:*

(*i*) $(x + 1)! = 12 \cdot (x - 1)!$ (*ii*) $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$

Solution. (*i*) We have $(x + 1)! = 12 \cdot (x - 1)!$

$\Rightarrow \quad (x + 1)x \cdot (x - 1)! = 12 \cdot (x - 1)!$

$\Rightarrow \quad (x + 1)x = 12 \qquad [\because \; (x - 1)! \neq 0]$

$\Rightarrow \quad x^2 + x - 12 = 0$

$\Rightarrow \quad x = -4, 3 \quad \therefore \quad x = \mathbf{3}. \qquad (\because \; -4 \notin \mathbf{N})$

(*ii*) We have $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$.

$\Rightarrow \quad \frac{1}{6!} + \frac{1}{7 \cdot 6!} = \frac{x}{8 \cdot 7 \cdot 6!} \Rightarrow \frac{1}{6!}\left(1 + \frac{1}{7}\right) = \frac{1}{6!}\left(\frac{x}{56}\right)$

$\Rightarrow \quad \frac{8}{7} = \frac{x}{56} \Rightarrow x = \mathbf{64}.$

20.3. PERMUTATIONS

An arrangement in a definite order of a number of things taking some or all at a time is called a **permutation.**

The total number of permutations of n distinct things taking $r\,(1 \le r \le n)$ at a time is denoted by ${}^nP_r$, where ${}^nP_r = \frac{n!}{(n-r)!}$, $0 \le r \le n$ or by $P(n, r)$. We define ${}^nP_0 = 1$.

where, ${}^nP_r = \frac{n!}{(n-r)!}$, $0 \le r \le n$.

Illustration. The permutations of 3 things *a, b, c* taking 2 at a time are:

ab	*bc*	*ca*
ba	*cb*	*ac.*

$\therefore \quad {}^3P_2 = 6$

The value of ${}^3P_2$ can also be found out by considering the problem of filling two places by using two out of *a, b, c.* Thus, the first place can be filled in 3 ways. After filling the first place, the second place can be filled in by using any of 2 (= 3 – 1) things.

$\therefore$ By **FPC**, the value of ${}^3P_2 = 3 \times 2 = 6$.

Working Rules

Rule I: ^nP_r (or $P(n, r)$) *denotes the number of permutations of n distinct things taking r at a time,* $1 \le r \le n$.

Rule II: *If value of r is given, then use:*

$$^nP_r = n(n-1)(n-2) \text{ .... } r \text{ factors.}$$

Rule III: *If value of r is not given, then use:* $^nP_r = \dfrac{n!}{(n-r)!}$.

Rule IV: $^nP_n = n! = n(n-1)(n-2) \text{ .... } 3 \cdot 2 \cdot 1.$

Example 5. *Evaluate:*

(*i*) $^5P_3$ (*ii*) $^7P_2$ (*iii*) $^{18}P_3$ (*iv*) $^6P_6$.

Solution. (*i*)

$$^5P_3 = \frac{5\,!}{(5-3)\,!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$= 5 \times 4 \times 3 = \mathbf{60}. \qquad \left(^nP_r = \frac{n\,!}{(n-r)\,!} \right)$$

(*ii*) $^7P_2 = 7 \times 6 = \mathbf{42}$.

(*iii*) $^{18}P_3 = 18 \times 17 \times 16 = \mathbf{4896}$.

(*iv*) $^6P_6 = \dfrac{6\,!}{(6-6)\,!} = \dfrac{6\,!}{0\,!} = \dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1} = \mathbf{720}$.

Example 6. *Show that:* $^{10}P_3 = {}^9P_3 + 3\,{}^9P_2$.

Solution.

$$\text{LHS} = {}^{10}P_3 = 10 \times 9 \times 8 = 720$$

$$\text{RHS} = {}^9P_3 + 3\,{}^9P_2 = (9 \times 8 \times 7) + 3(9 \times 8)$$

$$= 504 + 216 = 720$$

$$\therefore \quad \text{LHS} = \text{RHS}$$

Example 7. *Find n if:*

(*i*) $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}$, $n > 4$ (*ii*) $^{n-1}P_3 : {}^nP_4 = 1 : 9$.

Solution (*i*) We have $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}$.

$$\Rightarrow \quad \frac{n(n-1)(n-2)(n-3)}{(n-1)(n-1-1)(n-1-2)(n-1-3)} = \frac{5}{3} \quad \Rightarrow \quad \frac{n}{n-4} = \frac{5}{3}$$

$$\Rightarrow \quad 3n = 5n - 20 \quad \Rightarrow \quad 2n = 20 \quad \Rightarrow \quad n = \mathbf{10}.$$

(*ii*) We have ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.

$$\Rightarrow \quad \frac{{}^{n-1}P_3}{{}^{n}P_4} = \frac{1}{9} \quad \Rightarrow \quad \frac{(n-1)(n-1-1)(n-1-2)}{n(n-1)(n-2)(n-3)} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{1}{n} = \frac{1}{9} \quad \Rightarrow \quad n = \mathbf{9}.$$

20.4. COMBINATIONS

A selection (group) of a number of things taking some or all at a time is called a **combination**.

The total number of combinations of n distinct things taking $r(1 \le r \le n)$ at a time is denoted by ${}^nC_r$ or by $C(n, r)$. We define ${}^nC_0 = 1$.

where $${}^nC_r = \frac{n!}{r!(n-r)!},\ 0 \le r \le n$$

Illustration. The combinations of 3 things a, b, c taking 2 at a time are:

$ab \qquad bc \qquad ca.$

$$\therefore \quad {}^3C_2 = 3$$

Working Rules

Rule I *${}^nC_r$ (or $C(n, r)$) denotes the number of combinations of n distinct things taking r at a time, $1 \le r \le n$.*

Rule II *If value of r is given, then use:*

$${}^nC_r = \frac{n(n-1)(n-2)......r \text{ factors}}{1 \cdot 2 \cdot 3 \,......r}$$

Rule III *If value of r is not given, then use:* ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

Rule IV (*i*) ${}^nC_r = {}^nC_{n-r}$

(*ii*) *If* ${}^nC_p = {}^nC_q$, *then either* $p = q$ *or* $p + q = n$.

Rule V ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r,\ 1 \le r \le n$.

Rule VI ${}^nC_0 = 1$ *and* ${}^nC_n = 1$.

Example 8. *Evaluate the following:*

(*i*) ${}^9C_4$ (*ii*) ${}^{51}C_{49}$.

Solution. (*i*) $${}^9C_4 = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} = \mathbf{126}.$$

Alternative method

$$^9C_4 = \frac{9\times 8\times 7\times 6}{1\times 2\times 3\times 4} = \mathbf{126}. \qquad \left(\because \ {}^nC_r = \frac{n(n-1)\ldots\ldots r \text{ factors}}{1\times 2\times\ldots\ldots\times r}\right)$$

(*ii*) $$^{51}C_{49} = {}^{51}C_{51-49} = {}^{51}C_2 = \frac{51\times 50}{1\times 2} = \mathbf{1275}. \qquad (\because \ {}^nC_r = {}^nC_{n-r})$$

20.5. REVIEW BASIC CONCEPT SETS

I. Finite and Infinite Sets

A set is said to be a **finite set** if it contains only finite number of elements. Otherwise, the set is said to be an **infinite set.**

If A is a finite set, then the number of elements in A is denoted by n(A) and called the **cardinality** of the finite set A.

Illustrations

1. The set {2, 4, 5, 10} is a finite set, because it contains only 4 elements.

2. The set of all even numbers is an infinite set.

II. Null Set

A set is said to be a **null set** if it does not contain any element. A null set is also called as *empty set* or *void set*. A null set is denoted by ϕ.

$$\therefore \qquad \phi = \{\,\}$$

Illustration 1. The set {0} is not a null set, because this set contains one element, namely '0'.

Illustration 2. Let A = $\{x : x \in \mathbf{N}, 2 < x < 3\}$. A does not contain any element, because there is no natural number between 2 and 3.

III. Singleton Set

A set is said to be a **singleton set** if it contains only one element.

Illustration 1 The sets {7}, {–15} are singleton sets.

Illustration 2 $\{x : x + 4 = 0, x \in \mathbf{Z}\}$ is a singleton set, because this set contains only one integer namely, –4.

Example 9. *Which of the following sets are singleton sets?*

(*i*) $\{x : x^2 + 2x + 1 = 0, x \in \mathbf{N}\}$ (*ii*) $\{x : x^2 = 9, |x| \leq 3, x \in \mathbf{N}\}$.

Solution (*i*) $x^2 + 2x + 1 = 0 \Rightarrow (x + 1)^2 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$

$x = -1$ is not a natural number.

$\therefore$ Given set = { }. This is not a singleton set.

(*ii*) $x^2 = 9$ implies $x = \pm 3$. $|x| \leq 3$ implies $x = -3, -2, -1, 0, 1, 2, 3$

$\therefore$ $x = -3, 3$ satisfy $x^2 = 9$ and $|x| \leq 3$ both.

Out of -3, 3, only $3 \in \mathbf{N}$.

$\therefore$ Given set = {3}. This is a singleton set.

IV. Equivalent Sets

Two sets A and B are said to be **equivalent sets** if the elements of A can be paired with the elements of B so that to each element of A there corresponds exactly one element of B and to each element of B there corresponds exactly one element of A.

Remark. Finite sets *A* and *B* are equivalent if and only if the number of elements in *A* and *B* are equal.

Illustration 1. The sets {*a*, *b*, *c*} and {4, 7, 10} are equivalent sets.

Illustration 2. The sets {1, 2, 3, 4, 5, 6, ...} and {2, 4, 6, 8, 10, 12, ...} are equivalent sets because of the correspondence:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \text{- - - - - - -} \\ 2 & 4 & 6 & 8 & \end{array}$$

V. Equal Sets

Two sets are said to be **equal sets** if every element of one set is in the other set and *vice versa*.

In other words, the sets A and B are equal, if

$x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

If sets A and B are equal, then we write A = B.

Illustrations

1. Let A = {2, 3, 5, 6} and B = {2, 3, 4, 5, 6}. We have $A \neq B$, because $4 \in B$ and $4 \notin A$.

2. Let A = {2, 3, 4} and B = {2, 3, 4, 4, 4}. We have A = B.

Note. It is sufficient to write an element in a set only once.

3. Let A = {1, 4, 5} and B = {4, 1, 5}. We have A = B.

Note. Order of elements in a set is immaterial.

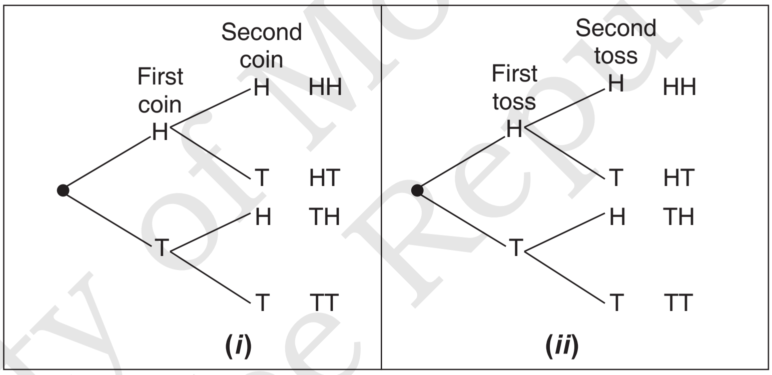
Remark. Equal sets are equivalent sets but equivalent sets may not be equal sets. For example, {2, 4, 5} and {3, 7, 8} are equivalent sets but not equal sets.

II. Tree Diagram

A **tree diagram** is a device used to enumerate all the logical possibilities of a sequence of events where each event can occur in a finite number of ways. A tree diagram is constructed from left to right and the number of branches at any point corresponds to the number of ways the next event can occur.

Illustration 1 The tree diagram of the sample space of the toss of two coins is shown in Fig. (*i*).

Illustration 2 The tree diagram of the sample space of the two tosses of a coin is shown in Fig. (*ii*).



(*i*) (*ii*)

Example 10. *A bag contains 4 red balls. What is the sample space if the random experiment consists of choosing:*

(*i*) *1 ball* (*ii*) *2 balls* (*iii*) *3 balls* (*iv*) *4 balls?*

Solution. Let the red balls be denoted by R_1, R_2, R_3 and R_4.

(*i*) In this experiment, one ball is drawn.

Number of elements in $S = {}^4C_1 = 4$

$\therefore$ $S = \{\mathbf{R_1, R_2, R_3, R_4}\}$.

(*ii*) In this experiment, two balls are drawn.

Number of elements in $S = {}^4C_2 = \dfrac{4 \times 3}{1 \times 2} = 6$

$\therefore$ $S = \{\mathbf{R_1R_2, R_2R_3, R_3R_4, R_4R_1, R_1R_3, R_2R_4}\}$.

(*iii*) In this experiment, three balls are drawn.

Number of elements in $S = {}^4C_3 = \dfrac{4 \times 3 \times 2}{1 \times 2 \times 3} = 4$

$\therefore$ $S = \{\mathbf{R_1R_2R_3, R_2R_3R_4, R_3R_4R_1, R_4R_1R_2}\}$.

(*iv*) In this experiment, 4 balls are drawn.

Number of elements in $S = {}^4C_4 = 1$

$\therefore$ $S = \{\mathbf{R_1R_2R_3R_4}\}$.

20.6. SET OPERATIONS

I. Venn Diagrams

The relationships between sets can be easily visualized by means of diagrams called **Venn diagrams.** Venn diagrams are named after the English logician *John Venn* (1834–1883).

II. Universal Set

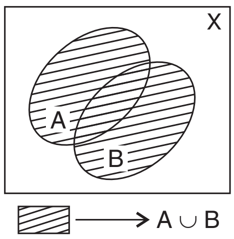
A set X is called a **universal set** if every set under consideration is a subset of X. The universal set is not a fixed set. It varies from situation to situation. A universal set is also denoted by 'U'.

In Venn diagrams, a universal set is depicted by the interior of a rectangle, whereas the subsets of universal set are depicted by the interior of circles, ellipses etc.

Illustration. The sets {1, 4}, {2, 3, 4}, {10, 19, 25} may be considered as subsets of universal set X = **N.** Here we may also take X = **Z**.

III. Union of Sets

The **union** of two sets A and B is defined as the set of all those elements which are in either A or B or both. The union of sets A and B is denoted as A ∪ B. The symbol '∪' is used to denote the 'union'.



In symbols, we write

$$\mathbf{A \cup B = \{x : x \in A \text{ or } x \in B\}}.$$

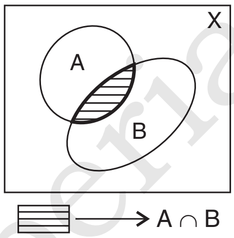
Illustrations

1. If A = {1, 2, 3, 4} and B = {2, 3, 4, 5, 6}, then

A ∪ B = {1, 2, 3, 4, 5, 6}.

IV. Intersection of Sets

The **intersection** of two sets A and B is defined as the set of all those elements which are in both A and B. The intersection of A and B is denoted by A ∩ B. The symbol '∩' is used to denote the 'intersection'.



In symbols, we write

$$\mathbf{A \cap B = \{x : x \in A \text{ and } x \in B\}}.$$

Illustrations

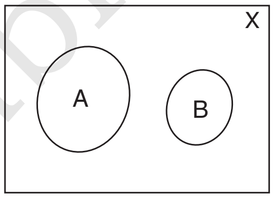
1. If A = {1, 2, 3, 4} and B = {2, 3, 4, 5, 6}, then A ∩ B = {2, 3, 4}.

2. If A = $\{x : x \in \mathbf{N},\ 0 < x < 5\}$ and B = $\{x : x \in \mathbf{N},\ 4 < x < 6\}$, then

$$A \cap B = \{1, 2, 3, 4\} \cap \{5\} = \phi.$$

V. Disjoint Sets

Two sets A and B are said to be **disjoint sets** if there is no element which belongs to both A and B. If A and B are disjoint sets, then A ∩ B = φ.

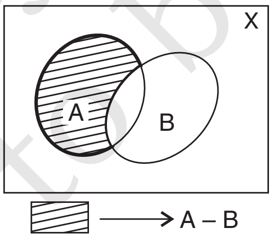


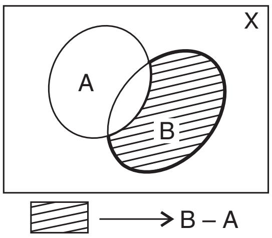
A ∩ B = φ

Illustration 1 The sets A = {4, 6, 10} and B = {7, 11, 15} are disjoint sets.

VI. Difference of Sets

The **difference** of two sets A and B in this order is the set of all those elements of A which are not in B. The difference of A and B in this order is denoted by A – B.





In symbols, we write **A – B = {x : x ∈ A and x ∉ B}**

Similarly, B – A = $\{x : x \in B \text{ and } x \notin A\}$.

Remark. In general, A – B and B – A are not equal sets. These are always disjoint sets.

Example 11. *Find A – B and B – A if:*

(*i*) *A* = {*2, 3, 6*}, *B* = {*1, 3, 7, 10*} (*ii*) *A* = {*3, 6, 7*}, *B* = {*1, 2, 5, 8*}.

Solution. (*i*) A – B = {2, 3, 6} – {1, 3, 7, 10} = **{2, 6}**

and B – A = {1, 3, 7, 10} – {2, 3, 6} = **{1, 7, 10}**

(*ii*) A – B = {3, 6, 7} – {1, 2, 5, 8} = **{3, 6, 7}**

and *B – A* = {1, 2, 5, 8} – {3, 6, 7} = **{1, 2, 5, 8}**.

Example 12. *If A* = {*4, 5, 8, 12*}, *B* = {*1, 4, 6, 9*} *and C* = {*1, 2, 4, 7, 8, 10*}, *then find*:

(*i*) *A* – (*B* – *A*) (*ii*) *A* – (*C* – *B*).

Solution. (*i*) B – A = {1, 4, 6, 9} – {4, 5, 8, 12} = {1, 6, 9}

∴ A – (B – A) = {4, 5, 8, 12} – {1, 6, 9} = **{4, 5, 8, 12}**

(*ii*) C – B = {1, 2, 4, 7, 8, 10} – {1, 4, 6, 9} = {2, 7, 8, 10}

∴ A – (C – B) = {4, 5, 8, 12} – {2, 7, 8, 10} = **{4, 5, 12}**.

20.7. CONTIGENCY TABLE

Contigency tables (also called crosstable or two-way tables) are used in statistics to summarize the relationship between several categorical variables. A contigency table is a special type of frequency distribution table, where two variables as shown simultaneously.

Example:

	Pizza rolls	***Chips and Dip***	***Cookies***	***Totals***
Poker	10	3	12	25
Trivial pursuit	8	14	7	29
Monopoly 14	17	7	38	
WE Bowling	12	7	4	23
Totals	44	41	30	115

20.8. SAMPLE SPACE

The **sample space** of a random experiment is defined as the set of all possible outcomes of the experiment. The possible outcomes *i.e.,* the elements of the sample space are called **sample points.** The sample space is generally denoted by the letter *S*.

We list the sample space of some random experiments.

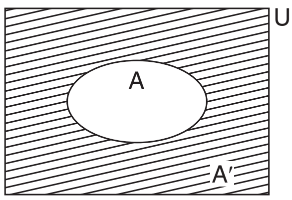
Random Experiment	Sample Space
1. Tossing of an unbiased coin	S = {H, T}
2. Tossing of an unbiased coin twice	S = {HH, HT, TH, TT} In *S*, the sample point *HT* represents 'head' on first toss and tail on second toss.
3. Tossing of two unbiased coins	S = {HH, HT, TH, TT} In S, the sample point HT represents 'head' on first coin and 'tail' on second coin.
4. Tossing of three unbiased coins	S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}*

Probability Event

An event is a collection of sample points with a common property. It is a subset of the sample space. For example, when a die is cast, the event E of throwing an even number is E = {2, 4, 6}. Also when two coins are tossed, the event of getting exactly one head E = {HT, TH}.

Complement of an Event

Let S be the sample space and A be an event of S. The se of all those sample points which are in S but not in A is called the complement of the event A and is denoted by $\bar{A}$. See figure. Note that the complement of a certain event is an impossible event.



Compound Events

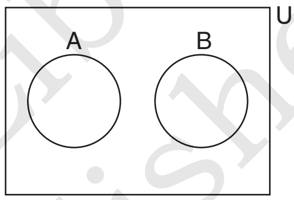
Events can be combined by the words 'or' and 'and'. Events which are thus combined are called compounds events. The words 'or' and 'and' correspond to '$\cup$' and '$\cap$' respectively in sets. Let A and B be two events. The compound events:

1. A ∪ B (*i.e.,* A or B) means eitherevent A or even B occur or both events occur.
2. A ∩ B (*i.e.,* A and B) means the events A and B occur together.

Note that "**or**" in probability means *addition*, "**and**" means multiplication

Mutually Exclusive Events

If A and B are events such that the two events cannot occur together or at the same time, then we say that the events A and B are **mutually exclusive** events. Thus if A ∩ B = ∅, then events A and B are mutually exclusive (see figure). For example, when you toss a coin, you can get either a head or a tail, not both at the same time. So the events of *getting a head* and *getting a tail* are mutually exclusive.



In this case, A and B are disjoint sets.

Independent Events

Two events A and B, are said to be independent if the occurrence of A does not affect the occurrence of B and vice versa. For example, if a fair die is thrown twice; let the event A = throwing a six on the first throw and B = throwing a siz on the second throw. The events A and B have no influence over each other and are therefore said to be independent.

What is a Conditional Event?

A conditional event algebra (CEA) contains not just ordinary events but alos conditional events, which have the form "if A, then B, The usual purpose of a CEA is to enable the defining of a probability function P, that satisfies the equation P(if A then B) = P(A and B)/P(A).

From now onward, we shall always assume that the outcomes of any given random experiment are *equally likely* unless the contrary is stated explicitely.

Example 13. *Three coins are tossed simultaneously. Write the sample space and the probabilities of getting (i) no head and (ii) two heads.*

Solution Here $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

(*i*) $P(\text{no head}) = P(\{TTT\}) = \dfrac{n(\{TTT\})}{n(S)} = \dfrac{1}{8}$.

(*ii*) $P(\text{two heads}) = P(\{HHT, HTH, THH\}) = \frac{3}{8}$.

20.9. 'ODDS INFAVOUR' AND 'ODDS AGAINST' AN EVENT

Let E be an event of a random experiment.

The ratio $P(E) : P(\bar{E})$ is called the **odds infavour** of happening of the event E.

The ratio $P(\bar{E}) : P(E)$ is called the **odds against** the happening of the event E.

Let odds in favour of an event E be $m : n$.

Let $P(E) = p$. $\therefore$ $P(\bar{E}) = 1 - p$

$\therefore$ $P(E) : P(\bar{E}) = m : n$ $\Rightarrow$ $p : 1 - p = m : n$

$\Rightarrow$ $\frac{p}{1-p} = \frac{m}{n}$ $\Rightarrow$ $np = m - mp$

$\Rightarrow$ $p = \frac{m}{m+n}$ *i.e.*, $P(E) = \frac{m}{m+n}$.

$\therefore$ **If odds infavour of E are $m : n$, then $P(E) = \frac{m}{m+n}$.**

Similarly, if odds against E are $m : n$, then odds infavour of E are $n : m$ and we have $P(E) = \frac{n}{n+m}$.

Example 14. *Find the probability of the event A if* (*i*) *odds infavour of event A are 5 : 7* (*ii*) *odds against A are 3 : 4.*

Solution. (*i*) Odds infavour of event A are 5 : 7.

Let $P(A) = p$. $\therefore$ $P(\bar{A}) = 1 - p$

$\therefore$ $p : 1 - p = 5 : 7$

$\Rightarrow$ $\frac{p}{1-p} = \frac{5}{7}$ $\Rightarrow$ $7p = 5 - 5p$

$\Rightarrow$ $12p = 5$ $\Rightarrow$ $p = \frac{5}{12}$

$\therefore$ $P(A) = \mathbf{\frac{5}{12}}$.

(*ii*) Odds against event A are 3 : 4. Let $P(A) = p$. $\therefore$ $P(\bar{A}) = 1 - p$

$\therefore$ $1 - p : p = 3 : 4$

$$\Rightarrow \quad \frac{1-p}{p} = \frac{3}{4} \quad \Rightarrow \quad 4 - 4p = 3p$$

$$\Rightarrow \quad 7p = 4 \quad \Rightarrow \quad p = \frac{4}{7}$$

$$\therefore \quad P(A) = \mathbf{\frac{4}{7}}.$$

Expected Value

Suppose the random variable x can take on the n values x_1, x_2, ... x_n. Also suppose the probabilities that these values occur are respectively p_1, p_2, ..., p_n. Then the expected value of the random variable is:

$$E(x) = x_1p_1 + x_2p_2 + \dots + x_np_n$$

Measurse of Discrete Random Variables

Expected value of a discrete distribution

- An weighted average, taking into account the probability
- The expected value of random varibale x is denoted as $E(x)$

$$E(x) = \Sigma xiP(xi)$$

$$= x_1P(x_1) + x_2P(x_2) + \dots + x_nP(x_n)$$

Example 15. *What is your expected gain when you play the flip-coin game twice?*

x	$P(x)$
–2	.25
0	.050
2	.25

$$E(x) = (-2) * 0.25 + 0 * 0.5 + 2 * 0.25$$
$$= 0$$

Your expected gain is 0! – a fair game.

EXERCISE

1. In a railway compartment, 6 seats are vacant on a beach. In how many ways can 3 passenegers sit on them?
2. Find the number of evetn positive numbers which have three digits.
3. Show that $n! + 1$ is not divisible by any number from 2 to n.

4. Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?

5. Find n if:

(*i*) $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}$, $n > 4$ (*ii*) $^{n-1}P_3 : {}^nP_4 = 1 : 9$.

6. Prove that: $^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$, $1 \le r \le n - 1$.

7. Find n, if $^{2n-1}P_n : {}^{2n+1}P_{n-1} = 22 : 7$.

8. Find n if:

(*i*) $^nC_8 = {}^nC_6$ (*ii*) $^nC_{n-4} = 5$

(*iii*) $^{25}C_{n+5} = {}^{25}C_{2n-1}$ (*iv*) $^{2n}C_3 : {}^nC_3 = 12 : 1$.

9. Find the value of:

(*i*) $^{15}C_3 + {}^{15}C_2$ (*ii*) $^{18}C_6 - {}^{17}C_6$

10. If $^nC_4$, $^nC_5$, $^nC_6$ are in A.P., find the value of n.

11. Which of the following sets are null sets?

(*i*) The set A of all prime numbers lying between 15 and 19

(*ii*) A = $\{x : x < 5, x > 6\}$ (*iii*) A = $\{x : x^2 = 16, x \in \mathbf{N}\}$.

12. Which of the following statements are true?

(*i*) If A = $\{x : x^2 = 4, x \in \mathbf{N}\}$, B = $\{-2\}$, then A ≠ B.

(*ii*) If A = $\{x : |x| < 2, x \in \mathbf{Z}\}$, B = $\{-1, 1\}$, then A = B.

(*iii*) If A = {1, 2, 3, 4, 5, 5}, B = {2, 1, 3, 4, 5}, then A = B.

(*iv*) If A = $\{x : x^2 - 5x + 7 = 0, x \in \mathbf{R}\}$, B = ϕ, then A = B.

13. The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.

14. A die is thrown repeatedly untill a six comes up. What is the sample space for this experiment?

15. Find A ∪ B if:

(*i*) A = {1, 2, 5}, B = {2, 3, 5, 7, 9} (*ii*) A = {3, 4, 7}, B = {1, 5, 6, 8}

(*iii*) A = ϕ, B = {2, 6, 8} (*iv*) A = {6, 7}, B = {1, 5, 6, 7, 9}.

16. Find A ∩ B if:

(*i*) A = {2, 3, 6}, B = {1, 3, 4, 6, 8}

(*ii*) A = {2, 3, 4, 8}, B = {1, 6}

(*iii*) A = ϕ, B = {3, 8, 11} (*iv*) A = {1, 3, 5}, B = {1, 2, 3, 4, 5, 6}.